

Lecture 7: October 19, 2026

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1 Singular Value Decomposition for matrices

Using the previous discussion, we can write matrices in convenient form. Let $A \in \mathbb{C}^{m \times n}$, which can be thought of as an operator from $\mathbb{C}^n$ to $\mathbb{C}^m$. Let $\sigma_1, \dots, \sigma_r$ be the non-zero singular values and let $v_1, \dots, v_r$ and $w_1, \dots, w_r$ be the right and left singular vectors respectively. Note that $V = \mathbb{C}^n$ and $W = \mathbb{C}^m$ and $v \in V, w \in W$, we can write the operator $|w\rangle\langle v|$ as the matrix wv^* , where v^* denotes $\overline{v^T}$. This is because for any $u \in V$, $wv^*u = w(v^*u) = \langle v, u \rangle \cdot w$. Thus, we can write

$$A = \sum_{i=1}^r \sigma_i \cdot w_i v_i^*.$$

Let $W \in \mathbb{C}^{m \times r}$ be a matrix with $w_1, \dots, w_r$ as columns, such that i^{th} column equals w_i . Similarly, let $V \in \mathbb{C}^{n \times r}$ be a matrix with $v_1, \dots, v_r$ as the columns. Let $\Sigma \in \mathbb{C}^{r \times r}$ be a diagonal matrix with $\Sigma_{ii} = \sigma_i$. Then, check that the above expression for A can also be written as

$$A = W \Sigma V^*$$

$$= \begin{pmatrix} \begin{array}{c} w_1 \\ \vdots \end{array} & \dots & \begin{array}{c} w_r \\ \vdots \end{array} \end{pmatrix} \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{pmatrix} \begin{pmatrix} \begin{array}{c} \text{---} \\ \vdots \\ \text{---} \end{array} v_1^* \\ \vdots \\ \begin{array}{c} \text{---} \\ \vdots \\ \text{---} \end{array} v_r^* \end{pmatrix}$$

where $V^* = \overline{V^T}$ as before.

We can also complete the bases $\{v_1, \dots, v_r\}$ and $\{w_1, \dots, w_r\}$ to bases for $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively and write the above in terms of unitary matrices.

Definition 1.1 A matrix $U \in \mathbb{C}^{n \times n}$ is known as a unitary matrix if the columns of U form an orthonormal basis for $\mathbb{C}^n$.

Proposition 1.2 Let $U \in \mathbb{C}^{n \times n}$ be a unitary matrix. Then $UU^* = U^*U = \text{id}$, where id denotes the identity matrix.

Let $\{v_1, \dots, v_n\}$ be a completion of $\{v_1, \dots, v_r\}$ to an orthonormal basis of $\mathbb{C}^n$, and let $V_n \in \mathbb{C}^{n \times n}$ be a unitary matrix with $\{v_1, \dots, v_n\}$ as columns. Similarly, let $W_m \in \mathbb{C}^{m \times m}$ be a unitary matrix with a completion of $\{w_1, \dots, w_r\}$ as columns. Let $\Sigma' \in \mathbb{C}^{m \times n}$ be a matrix with $\Sigma'_{ii} = \sigma_i$ if $i \leq r$, and all other entries equal to zero. Then, we can also write

$$A = W_m \Sigma' V_n^* = \begin{pmatrix} \begin{matrix} w_1 \\ \vdots \\ w_r \end{matrix} & \begin{matrix} w_{r+1} \\ \vdots \\ w_m \end{matrix} \end{pmatrix} \begin{pmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ & & & 0 & \ddots \\ & & & & & 0 \end{pmatrix} \begin{pmatrix} \text{---} \\ \vdots \\ \text{---} \\ \vdots \\ \text{---} \end{pmatrix} \begin{pmatrix} v_1^* \\ \vdots \\ v_r^* \\ v_{r+1}^* \\ \vdots \\ v_n^* \end{pmatrix}$$

2 Low-rank approximation for matrices

Given a matrix $A \in \mathbb{C}^{m \times n}$, we want to find a matrix B of rank at most k which “approximates” A . For now we will consider the notion of approximation in spectral norm i.e., we want to minimize $\|A - B\|_2$, where

$$\|(A - B)\|_2 = \max_{v \neq 0} \frac{\|(A - B)v\|_2}{\|v\|_2}.$$

Here, $\|v\|_2 = \sqrt{\langle v, v \rangle}$ denotes the norm defined by the standard inner product on $\mathbb{C}^n$. The 2 in the notation $\|\cdot\|_2$ comes from the expression we get by expressing v in the orthonormal basis of the coordinate vectors. If $v = (c_1, \dots, c_n)^T$, then $\|v\|_2 = \left(\sum_{i=1}^n |c_i|^2\right)^{1/2}$ which is simply the Euclidean norm we are familiar with¹. Note that while the norm here seems

¹In general, one can consider the norm $\|v\|_p := \left(\sum_{i=1}^n |c_i|^p\right)^{1/p}$ for any $p \geq 1$. While these are indeed valid notions of distance satisfying a triangle inequality for any $p \geq 1$, they do not arise as a square root of an inner product when $p \neq 2$.

to be defined in terms of the coefficients, which indeed depend on the choice of the orthonormal basis, the value of the norm is in fact $\sqrt{\langle v, v \rangle}$ which is just a function of the vector itself and not of the basis we are working with. The basis and the coefficients merely provide a convenient way of computing the norm.

SVD also gives the optimal solution for another notion of approximation: minimizing the Frobenius norm $\|A - B\|_F$, which equals $(\sum_{ij} (A_{ij} - B_{ij})^2)^{1/2}$. We will see this later. Let $A = \sum_{i=1}^r w_i v_i^*$ be the singular value decomposition of A and let $\sigma_1 \geq \dots \geq \sigma_r > 0$. If $k \geq r$, we can simply use $B = A$ since $\text{rank}(A) = r$. If $k < r$, we claim that $A_k = \sum_{i=1}^k \sigma_i w_i v_i^*$ is the optimal solution.

Proposition 2.1 $\|A - A_k\|_2 = \sigma_{k+1}$.

Proof: Complete $v_1, \dots, v_k$ to an orthonormal basis $v_1, \dots, v_n$ for $\mathbb{C}^n$. Given any $v \in \mathbb{C}^n$, we can uniquely express it as $\sum_{i=1}^n c_i \cdot v_i$ for appropriate coefficients $c_1, \dots, c_n$. Thus, we have

$$(A - A_k)v = \left(\sum_{j=k+1}^r \sigma_j \cdot w_j v_j^* \right) \left(\sum_{i=1}^n c_i \cdot v_i \right) = \sum_{j=k+1}^r \sum_{i=1}^n c_i \sigma_j \cdot \langle v_j, v_i \rangle \cdot w_j = \sum_{j=k+1}^r c_j \sigma_j \cdot w_j,$$

where the last equality uses the orthonormality of $\{v_1, \dots, v_n\}$. We can also complete $w_1, \dots, w_r$ to an orthonormal basis $w_1, \dots, w_m$ for $\mathbb{C}^m$. Since $(A - A_k)$ is already expressed in this basis above, we get that

$$\|(A - A_k)v\|_2^2 = \left\| \sum_{j=k+1}^r c_j \sigma_j \cdot w_j \right\|_2^2 = \left\langle \sum_{j=k+1}^r c_j \sigma_j \cdot w_j, \sum_{j=k+1}^r c_j \sigma_j \cdot w_j \right\rangle = \sum_{j=k+1}^r |c_j|^2 \cdot \sigma_j^2.$$

Finally, as in the computation with Rayleigh quotients, we have that for any $v \neq 0$ expressed as $v = \sum_{i=1}^n c_i \cdot v_i$,

$$\frac{\|(A - A_k)v\|_2^2}{\|v\|_2^2} = \frac{\sum_{j=k+1}^r |c_j|^2 \cdot \sigma_j^2}{\sum_{i=1}^n |c_i|^2} \leq \frac{\sum_{j=k+1}^r |c_j|^2 \cdot \sigma_{k+1}^2}{\sum_{i=1}^n |c_i|^2} \leq \sigma_{k+1}^2.$$

This gives that $\|A - A_k\|_2 \leq \sigma_{k+1}$. Check that it is in fact equal to σ_{k+1} (why?) ■

In fact the proof above actually shows the following:

Exercise 2.2 Let $M \in \mathbb{C}^{m \times n}$ be any matrix with singular values $\sigma_1 \geq \dots \geq \sigma_r > 0$. Then, $\|M\|_2 = \sigma_1$ i.e., the spectral norm of a matrix is actually equal to its largest singular value.

Thus, we know that the error of the best approximation B is at most σ_{k+1} . To show the lower bound, we need the following fact.

Exercise 2.3 Let V be a finite-dimensional vector space and let S_1, S_2 be subspaces of V . Then, $S_1 \cap S_2$ is also a subspace and satisfies

$$\dim(S_1 \cap S_2) \geq \dim(S_1) + \dim(S_2) - \dim(V).$$

We can now show the following.

Proposition 2.4 Let $B \in \mathbb{C}^{m \times n}$ have $\text{rank}(B) \leq k$ and let $k < r$. Then $\|A - B\|_2 \geq \sigma_{k+1}$.

Proof: By rank-nullity theorem $\dim(\ker(B)) \geq n - k$. Thus, by the fact above

$$\dim(\ker(B) \cap \text{Span}(v_1, \dots, v_{k+1})) \geq (n - k) + (k + 1) - n \geq 1.$$

Thus, there exists a $z \in \ker(B) \cap \text{Span}(v_1, \dots, v_{k+1}) \setminus \{0\}$. Then,

$$\begin{aligned} \|(A - B)z\|_2^2 &= \|Az\|_2^2 = \langle z, A^*Az \rangle = \mathcal{R}_{A^*A}(z) \cdot \|z\|_2^2 \\ &\geq \left(\min_{y \in \text{Span}(v_1, \dots, v_{k+1}) \setminus \{0\}} \mathcal{R}_{A^*A}(y) \right) \cdot \|z\|_2^2 \\ &\geq \sigma_{k+1}^2 \cdot \|z\|_2^2. \end{aligned}$$

Thus, there exists a $z \neq 0$ such that $\|(A - B)z\|_2 \geq \sigma_{k+1} \cdot \|z\|_2$, which implies $\|A - B\|_2 \geq \sigma_{k+1}$. ■